AMS 241: Bayesian Nonparametric Methods (Spring 2018)

Homework set on Dirichlet processes: prior properties and simulation
(due Tuesday April 24)

1. Assume a Dirichlet process (DP) prior, $\mathrm{DP}(\alpha, G_0)$, for distributions G on $\mathcal{X}$. Show that for any specified (measurable) disjoint subsets B_1 and B_2 of $\mathcal{X}$, $\mathrm{Corr}(G(B_1), G(B_2))$ is negative. Is the negative correlation for random probabilities induced by the DP prior a restriction? Discuss.

2. Simulation of Dirichlet process prior realizations
Consider a $\mathrm{DP}(\alpha, G_0)$ prior over the space of distributions (equivalently c.d.f.s) G on $\mathbb{R}$, with $G_0 = \mathrm{N}(0, 1)$.

(a) Use both Ferguson's original definition and Sethuraman's constructive definition to generate (multiple) prior c.d.f. realizations from the $\mathrm{DP}(\alpha, \mathrm{N}(0, 1))$, for different values of α ranging from *small* to *large*.

(b) In addition to prior c.d.f. realizations, obtain, for each value of α, the corresponding prior distribution for the mean functional $\mu(G) = \int t \, \mathrm{d}G(t)$ and for the variance functional $\sigma^2(G) = \int t^2 \, \mathrm{d}G(t) - \{\int t \, \mathrm{d}G(t)\}^2$.
(Note that, because G_0 has finite first and second moments, both of the random variables $\mu(G)$ and $\sigma^2(G)$ take finite values almost surely. The third problem asks you to prove this result for the mean functional.)

(c) Finally, consider simulation under a mixture of DPs (MDP) prior, which extends the DP above by adding a gamma prior for α. Therefore, the MDP prior for G is defined such that, given α, $G \mid \alpha \sim \mathrm{DP}(\alpha, \mathrm{N}(0, 1))$. To simulate from the MDP, you can use either of the DP definitions given draws for α from its prior. You can work with 2-3 different gamma priors for α.

3. (Optional) Assume a DP prior, $\mathrm{DP}(\alpha, G_0)$, for distributions G on $\mathbb{R}$, such that G_0 has finite mean, that is, $\mu(G_0) = \int t \, \mathrm{d}G_0(t) < \infty$. Use the DP constructive definition to show that the random variable $\mu(G) = \int t \, \mathrm{d}G(t)$ is almost surely finite. Moreover, show that $\mathrm{E}(\mu(G)) = \mu(G_0)$.
(**Hint.** Check absolute convergence working with random variable $\mu^*(G) = \int |t| \, \mathrm{d}G(t)$, and recall that if an $\mathbb{R}^+$-valued random variable has finite mean, then it is almost surely finite. The monotone convergence theorem is the main tool needed for the proof.)