AMS 241: Bayesian Nonparametric Methods – Spring 2018
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Modes of convergence for sequences of random variables

Given a sequence of random variables $\{X_n : n \geq 1\}$ and a limiting random variable X, there are several ways to formulate convergence "$X_n \to X$ as $n \to \infty$". The following four definitions are commonly used to study limiting results for random variables and stochastic processes.

Almost sure convergence ($X_n \to^{\text{a.s.}} X$).
Let $\{X_n : n \geq 1\}$ and X be random variables defined on some probability space $(\Omega, \mathcal{F}, P)$. $\{X_n : n \geq 1\}$ converges almost surely to X if

$$P\left(\left\{\omega \in \Omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)\right\}\right) = 1.$$

Convergence in rth mean ($X_n \to^{r-\text{mean}} X$).
Let $\{X_n : n \geq 1\}$ and X be random variables defined on some probability space $(\Omega, \mathcal{F}, P)$. $\{X_n : n \geq 1\}$ converges in mean of order $r \geq 1$ (or in rth mean) to X if $\mathrm{E}(|X_n^r|) < \infty$ for all n, and

$$\lim_{n\to\infty} \mathrm{E}(|X_n - X|^r) = 0.$$

Convergence in probability ($X_n \to^{\text{p}} X$).
Let $\{X_n : n \geq 1\}$ and X be random variables defined on some probability space $(\Omega, \mathcal{F}, P)$. $\{X_n : n \geq 1\}$ converges in probability to X if for any $\epsilon > 0$,

$$\lim_{n\to\infty} P(\{\omega \in \Omega : |X_n(\omega) - X(\omega)| > \epsilon\}) = 0.$$

Convergence in distribution ($X_n \to^{\text{d}} X$).
Let $\{X_n : n \geq 1\}$ and X be random variables with distribution functions $\{F_n : n \geq 1\}$ and F, respectively. $\{X_n : n \geq 1\}$ converges in distribution to X if

$$\lim_{n\to\infty} F_n(x) = F(x),$$

for all points x at which F is continuous.

Note that the first three types of convergence require that X_n and X are all defined on the same probability space, as they include statements involving the (common) probability measure P. However, convergence in distribution applies to random variables defined possibly on different probability spaces, as it only involves the corresponding distribution functions.

It can be shown that:
Almost sure convergence implies convergence in probability.
Convergence in rth mean implies convergence in probability, for any $r \geq 1$.
Convergence in probability implies convergence in distribution.
Convergence in rth mean implies convergence in sth mean, for $r > s \geq 1$.
No other implications hold without further assumptions on $\{X_n : n \geq 1\}$ and/or X.

Convergence theorems for expectations

Monotone convergence theorem: Consider a countable sequence $\{X_n : n = 1, 2, ...\}$ of $\overline{\mathbb{R}}^+$–valued random variables defined on the same probability space $(\Omega, \mathcal{F}, P)$. Assume that the sequence is pointwise (or almost surely) increasing, that is, for all n, $X_n(\omega) \leq X_{n+1}(\omega)$ for all $\omega \in \Omega$ (or all ω in an event of probability 1). Denote by X the pointwise (or almost sure) limit of the sequence $\{X_n : n = 1, 2, ...\}$.
• Then, $\lim_{n\to\infty} \mathrm{E}(X_n) = \mathrm{E}(X)$.

Dominated convergence theorem: Consider a countable sequence $\{X_n : n = 1, 2, ...\}$ of $\overline{\mathbb{R}}$–valued random variables defined on the same probability space $(\Omega, \mathcal{F}, P)$. Assume there exists a random variable Y (also defined on $(\Omega, \mathcal{F}, P)$) such that $|X_n| \leq Y$, almost surely for all n, and $\mathrm{E}(Y) < \infty$.
• Then,

$$-\infty < \mathrm{E}(\liminf_{n\to\infty} X_n) \leq \liminf_{n\to\infty} \mathrm{E}(X_n) \leq \limsup_{n\to\infty} \mathrm{E}(X_n) \leq \mathrm{E}(\limsup_{n\to\infty} X_n) < \infty$$

In addition to the assumptions $|X_n| \leq Y$, almost surely for all n, and $\mathrm{E}(Y) < \infty$, assume that the sequence $\{X_n : n = 1, 2, ...\}$ converges almost surely to random variable X (also defined on $(\Omega, \mathcal{F}, P)$).
• Then, $\mathrm{E}(|X|) < \infty$, $\lim_{n\to\infty} \mathrm{E}(X_n) = \mathrm{E}(X)$, and $\lim_{n\to\infty} \mathrm{E}(|X_n - X|) = 0$.

Bounded convergence theorem: Consider a countable sequence $\{X_n : n = 1, 2, ...\}$ of $\overline{\mathbb{R}}$–valued random variables defined on the same probability space $(\Omega, \mathcal{F}, P)$. Assume that the sequence converges almost surely to random variable X (also defined on $(\Omega, \mathcal{F}, P)$) and that $|X_n| \leq M$, almost surely for all n, where M is a finite constant.
• Then, $\mathrm{E}(|X|) \leq M$, $\lim_{n\to\infty} \mathrm{E}(X_n) = \mathrm{E}(X)$, and $\lim_{n\to\infty} \mathrm{E}(|X_n - X|) = 0$.